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On the Fundamental Postulate of Tamisage.

BY RAYMOND K. MORLEY.

1. *Introduction.*

As is well known, Sylvester and Franklin in their computation of tables of groundforms of binary quantics by means of generating functions made use of the so-called Fundamental Postulate of Tamisage, which is briefly that there can not exist a groundform and a fundamental syzygy of the same degorder. They inferred the truth of this from its holding for all binary quantics of order not greater than six, but they were unable to prove it in general.* In 1882 J. Hammond showed that there is an exception to this postulate in the case of the seventhic of degorder 5.13.† This result was at once confirmed by Cayley‡ and generally accepted by mathematicians, but none the less it seems to have been viewed with suspicion by Sylvester and his followers for some time, or at any rate to have been regarded as a very remarkable and perhaps solitary exception. This view was strengthened by the fact that Sylvester in 1881 had succeeded in proving the non-existence of a groundform of the eighthic of degorder 10.4, where von Gall thought he had found an exception.§ Besides this, the seventhic was regarded || as somewhat exceptional anyway, inasmuch as Sylvester and Franklin were in this case alone unable to exhibit the Representative Form of the Generating Function with a finite numerator, owing to their supposing that there was no invariant of order $10k$ (k integer). This assumption was, as a matter of fact, due directly to the Fundamental Postulate, and of course indicated nothing with regard to it.

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* See Sylvester, AMERICAN JOURNAL, Vol. II, p. 230; Franklin, AMERICAN JOURNAL, Vol. III, p. 128.

† *Proc. Lond. Math. Soc.*, Vol. XIV, p. 85, and AMERICAN JOURNAL, Vol. V, p. 218.

‡ *Proc. Lond. Math. Soc.*, Vol. XIV, p. 88.

§ Von Gall, *Math. Ann.*, Vol. XVII, pp. 31, 139, 456; Sylvester, *Comptes Rendus*, Vol. XCIII, pp. 192, 365; also AMERICAN JOURNAL, Vol. IV, p. 62.

|| See *e. g.* Elliott, "Algebra of Quantics," pp. 181–182. He seems to be unaware of Hammond's work cited below.

In 1888 von Gall computed the complete system of groundforms of the binary seventhic by the method of Gordan,* completing the work begun by Gordan himself in the well-known "Formensystem Binärer Formen." Gordan's method fails if at all by an excess of groundforms, the only proof of the irreducibility of the forms being the failure of the investigator to reduce them. On the other hand, the effect of the Fundamental Postulate is to make the method of Sylvester fail if at all by defect. Von Gall found a number of cases in which he was unable to make his results agree with Sylvester's, among them the case already pointed out by Hammond of degorder 5.13. Von Gall states (*loc. cit.*, p. 335) that in correspondence with him Sylvester said that he (Sylvester) had succeeded in proving the existence of a groundform of degorder 5.13, and admitted the possibility of other exceptions to the Fundamental Postulate. Hammond in 1890† by a direct method proved the existence of one irreducible invariant of the binary seventhic of order 20 and of one of order 30, in part confirming von Gall's work in those cases. With the aid of this result he reduced the Generating Function of the seventhic to a Representative Form with a finite numerator. This reason for regarding the seventhic as inherently exceptional was thus proven groundless.

There remain a number of cases in which von Gall's results for the seventhic do not agree with Sylvester's besides the three already mentioned in which Hammond has confirmed von Gall. The fact that such experts as Gordan and von Gall have failed to find the forms in question reducible does not of course prove them so (witness the case of the eighthic referred to above), however strong a presumption it may make in that direction. It should perhaps be mentioned here that Stroh‡ has deduced a theoretic criterion for testing reducibility, but it is not always easy to apply in any given case.

In consideration, therefore, of the lack of absolute proof either way in the cases of disagreement between Sylvester's results and von Gall's, it has seemed worth while to discuss some of the simpler of these cases. Further, although of course a single exception disproves the Fundamental Postulate as thoroughly as more from a logical standpoint, yet in view of the possibility that the seventhic might in some way peculiar to itself cause exceptions it seems desirable to point out that exceptions to the Postulate are by no means confined to the seventhic.

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* *Math. Ann.*, Vol. XXXI, p. 318.

† *Math. Ann.*, Vol. XXXVI, p. 255.

‡ *Math. Ann.*, Vol. XXXIII, p. 106.

2. *The Binary Seventhic.*

The present section shows that in the four simplest cases, after that discussed by Hammond, von Gall's results are correct, that Sylvester's table is defective owing to further exceptions to the Fundamental Postulate.

The following table shows the differences between Sylvester's results and von Gall's.* In the squares containing one figure the results agree. Where two figures are given the upper one is von Gall's result, the lower Sylvester's.

ORDER IN VARIABLES.

DEGREE IN COEFFICIENTS.	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1								1								
2			1				1				1					
3				1		1		1		1		1				1
4	1				2		1		2		1				1	
5		1		2		2		2		2				1 0		
6			3		2		2		2				1 0			
7		3		2		4		2				1 0				
8	3		3		3		3				1 0					
9		3		5		2				1 0						
10			4		4 3				1 0							
11		5		3				1 0								
12	6		6				1 0									
13		7		1 0		1 0										
14	4				2 0											
15		3		1 0												
16	2		3 0													
17		2														
18	9		1 0													
19		2 0														
20	2 0															
22	3 1															
23		1 0														
25		1 0														
26	2 0															
30	1 0															

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* Von Gall, *Math. Ann.*, Vol. XXI, p. 336; Sylvester, AMERICAN JOURNAL, Vol. II, p. 230.

Before taking up the cases of disagreement it will be convenient to collect certain abbreviations. The system of groundforms of the binary seventhic adopted and the abbreviations for them are those used by von Gall.

Groundforms of Binary Seventhic.

$f = a_x^7$;
$l = l_x^2 = (ff)^6$; $k = k_x^6 = (ff)^4$; $H = H_x^{10} = (ff)^2$;
$r = r_x^3 = (fk)^5$; $\varepsilon = \varepsilon_x^9 = (fk)^2$; $\gamma = \gamma_x^{11} = (fk)$; $T = T_x^{15} = (fH)$;
$C = (ll)^2$; $p = p_x^4 = (kk)^4$; $\Delta = \Delta_x^8 = (kk)^2$; $\zeta = \zeta_x^{14} = (Hk)$;
$g = g_x^5 = (rk)^2$; $\eta = \eta_x^7 = (rk)$;
$\tau = \tau_x^2 = (rr)^2$; $\beta = \beta_x^8 = (kp)$;
$\alpha = \alpha_x = (pr)^3$;
$\lambda = \lambda_x^6 = (\tau k)$; $\mu = \mu_x^4 = (\tau k)^2$.

As is usual the index scheme $\begin{pmatrix} f_1 & f_2 & f_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix}$ will be used as abbreviation for the well-known Gordan series,*

$$\sum_i \frac{\binom{n_2 - \alpha_1 - \alpha_3}{i}\binom{\alpha_2}{i}}{\binom{n_1 + n_2 - 2\alpha_3 - i + 1}{i}} \left((f_1 f_2)^{\alpha_3 + i} f_3\right)^{\alpha_1 + \alpha_2 - i}$$
$$= \left(-1\right)^{\alpha_1} \sum_i \frac{\binom{n_3 - \alpha_1 - \alpha_2}{i}\binom{\alpha_3}{i}}{\binom{n_1 + n_2 - 2\alpha_3 - i + 1}{i}} \left((f_1 f_3)^{\alpha_2 + i} f_2\right)^{\alpha_1 + \alpha_3 - i},$$

where $n_1\, n_2\, n_3$ are the orders of $f_1\, f_2\, f_3$ respectively, and either

$$\alpha_1 = 0, \quad \alpha_2 + \alpha_3 \leqq n_1, \quad \alpha_3 \leqq n_2, \quad \alpha_2 \leqq n_3;$$

or

$$\alpha_2 + \alpha_3 = n_1, \quad \alpha_1 + \alpha_3 \leqq n_2, \quad \alpha_1 + \alpha_2 \leqq n_3.$$

Certain basic or elementary syzygies are of the greatest use in the investigation. All but the last two of them are taken from Stroh,† though the simpler ones were known previously.

$f_1\, f_2\, f_3\, f_4$ denote forms of order as great as the weight of the syzygy in which they occur, unless otherwise stated. t is a cubic form. q and q' are quadratic forms.

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* Gordan, "Formensystem," p. 11, III.

† Stroh, *Math. Ann.*, Vol. XXXVI, p. 271.

WEIGHT 1.

$$(f_1 f_2 f_3) = f_1 (f_2 f_3) + f_2 (f_3 f_1) + f_3 (f_1 f_2) = 0.$$

WEIGHT 2.

$$\begin{aligned}[f_1 f_2 f_3 f_4]_2 = 2 (f_1 f_3)(f_2 f_4) + f_1 f_2 (f_3 f_4)^2 + (f_1 f_2)^2 f_3 f_4 \\ - f_1 f_4 (f_2 f_3)^2 - (f_1 f_4)^2 f_2 f_3 = 0.\end{aligned}$$

(The sign of the third term as printed in Stroh, *loc. cit.*, is incorrect.)

WEIGHT 3.

$$\begin{aligned}[f_1 f_2 f_3 q]_3 = 2 (f_1 f_2)(f_3 q)^2 - 2 (f_1 q)(f_2 f_3)^2 + 2 (f_2 q)(f_1 f_3)^2 \\ - f_3 ((f_1 q)^2 f_2) + f_3 ((f_2 q)^2 f_1) + f_3 q (f_1 f_2)^3 = 0.\end{aligned}$$

This syzygy holds also when f_3 is a quadratic form.

WEIGHT 4.

$$\begin{aligned}[f_1 f_2 q q']_4 = 2 (f_1 f_2)^2 (q q')^2 - 2 (f_1 f_2)^3 (q q') + (f_1 f_2)^4 q q' \\ - (f_1 q)^2 (f_2 q')^2 - (f_1 q')^2 (f_1 q)^2 - 2 q ((f_2 q')^2 f_1)^2 \\ - 2 q' ((f_1 q)^2 f_2)^2 + f_1 (f_2, q q')^4 + f_2 (f_1, q q')^4 = 0.\end{aligned}$$

The two following syzygies also of weight 4 are not given by Stroh:

$$\begin{aligned}[f_1 f_2 t q]_4 = & - 3 f_1 ((f_2 q)^2 t)^2 - 3 f_2 ((f_1 q)^2 t)^2 - 2 t q (f_1 f_2)^4 \\ & - 6 (f_1 f_2)^2 (t q)^2 + 3 (f_1 t)^2 (f_2 q)^2 + 3 (f_1 q)^2 (f_2 t)^2 \\ & + 3 t ((f_1 q)^2 f_2)^2 + 3 t ((f_2 q)^2 f_1)^2 + 2 q ((f_1 t)^3 f_2) \\ & + 2 q ((f_2 t)^3 f_1) = 0,\end{aligned}$$

$$\begin{aligned}\{f_1 f_2 t q\}_4 = & \, 2 f_2 ((f_1 t)^3 q) - 3 f_2 ((f_1 q)^2 t)^2 - t q (f_1 f_2)^4 \\ & - 3 (f_1 f_2)^2 (t q)^2 + 3 (f_1 t)^2 (f_2 q)^2 - 2 (f_1 q)(f_2 t)^3 \\ & + 2 t ((f_1 q)^2 f_2)^2 + t ((f_2 q)^2 f_1)^2 = 0.\end{aligned}$$

These syzygies are most easily proven by a method due to Stroh.* Let the forms f_1, f_2, t, q be represented as powers of linear functions,

$$f_1 = (x + a y)^{n_1}, \quad f_2 = (x + b y)^{n_2}, \quad t = (x + c y)^3, \quad q = (x + d y)^2.$$

Then the $(i + 1)$th coefficient of f_1, say a_i, is represented by a^i; of f_2, b_i, by b^i; and so on. Form the transvectants of these forms and take the first coefficient, which will be denoted by a subscript 0 on the transvectant. Then

$$(f_1 f_2)_0^\lambda = \sum_i (-1)^i \binom{\lambda}{i} a_i b_{\lambda - i}$$

may be represented by

$$\sum_i (-1)^i \binom{\lambda}{i} a^i b^{\lambda - i} = (a - b)^\lambda.$$

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* Stroh, *Math. Ann.*, Vol. XXXVI, p. 273.

Further, if the index of the transvectant is the same as the order of the second form within it, it is clear that the second coefficient of the transvectant is obtained from the first merely by increasing the subscript of every a involved by unity. A similar process gives the third from the second and so on. For instance, if the transvectant with the subscript j denotes the $(j+1)$th coefficient of the transvectant,

$$(f_1 t)^3_j = \sum_i (-1)^i \binom{3}{i} a_{j+i}\, c_{3-i}\,.$$

Then the first coefficient of a transvectant like $((f_1 t)^3 f_2)^\mu$,

$$\begin{aligned}((f_1 t)^3 f_2)^\mu_0 &= \sum_j (-1)^j \binom{\mu}{j} (f_1 t)^3_j\, b_{\mu-j}\\ &= \sum_j (-1)^j \binom{\mu}{j} b_{\mu-j} \sum_i (-1)^i \binom{3}{i} a_{j+i}\, c_{3-i}\,,\end{aligned}$$

may be represented by

$$\sum_j \sum_i (-1)^{j+i} \binom{\mu}{j} \binom{3}{i} a^{j+i}\, b^{\mu-j}\, c^{3-i} = (a-c)^3 (a-b)^\mu\,.$$

So also for other transvectants of similar form.

Then the syzygy $[f_1 f_2 tq]_4 = 0$ may be represented thus:

$$\begin{aligned}&-3(b-d)^2(b-c)^2 - 3(a-d)^2(a-c)^2 - 2(a-b)^4 - 6(a-b)^2(c-d)^2\\ &+3(a-c)^2(b-d)^2 + 3(a-d)^2(b-c)^2 + 3(a-d)^2(a-b)^2\\ &+3(b-d)^2(b-a)^2 + 2(a-c)^3(a-b) + 2(b-c)^3(b-a) = 0.\end{aligned}$$

Now put

$$a-b=-b',\quad a-c=-c',\quad a-d=-d';$$

then

$$b-c=b'-c',\quad b-d=b'-d',\quad c-d=c'-d'.$$

After making these substitutions, which serve to express the syzygant as a function of three variables, drop the primes and the result is exactly what would result if $a=0$ in the original expression. It becomes, therefore,

$$\begin{aligned}&-3(b-d)^2(b-c)^2 - 3c^2d^2 - 2b^4 - 6b^2(c-d)^2 + 3c^2(b-d)^2\\ &+3d^2(b-c)^2 + 3b^2d^2 + 3b^2(b-d)^2 + 2bc^3 + 2b(b-c)^3 = 0,\end{aligned}$$

which is easily verified.

$\{f_1 f_2 tq\}_4 = 0$ is proven in a precisely similar way.

Take up now the cases of disagreement between Sylvester's results and von Gall's. The simplest case is that in which Hammond has proven that there is an exception to the Fundamental Postulate, namely of degorder 5.13. A brief analysis of it will be included here for completeness.

Von Gall shows one irreducible covariant of this degorder, namely $(Tl)^2$. There are four reducible covariants, $(1.7).(4.6)$,[*] $(2.2).(3.11)$, $(2.6).(3.7)$, $(2.10).(3.3)$, which are

$$f.(kl), \quad l.\gamma, \quad k.(fl), \quad H.r.$$

A computation of $\Delta(7, 5; 13)$[†] shows that there are four linearly independent covariants of degorder 5.13. If, therefore, there exists a syzygy among these four reducible covariants, there must be a groundform. Further, von Gall's system is certainly complete; therefore if there is a groundform, the one he exhibits must be irreducible. There is such a syzygy, namely,

$$(fkl) = f(kl) + k(lf) + l(fk) = 0;$$

or

$$f(kl) - k(fl) + l.\gamma = 0.$$

There are then in this case both a fundamental syzygy and a groundform, and consequently an exception to the Fundamental Postulate.

The next case in which the two sets of results differ is that of degorder 6.12· Von Gall exhibits one irreducible covariant, namely $(\zeta l)^2$, and it is easy to find that there are the following seven reducible covariants:

$$(1.7)(5.5)[2], \quad (2.2)(4.10), \quad (2.6)(4.6), \quad (3.3)(3.9), \quad (3.5)(3.7),$$
$$(1.7)(2.2)(3.3),$$

which are:

$$f.g, \quad f.(fl^2)^3, \quad l.(Hl), \quad k.(kl), \quad r.\varepsilon, \quad (fl)^2.(fl), \quad f.l.r.$$

A computation of $\Delta(7, 6; 12)$ shows that there are seven linearly independent covariants of degorder 6.12. If, therefore, there exists a syzygy of this degorder, there must be a groundform, and by the same reasoning as above it may be taken as the form $(\zeta l)^2$. There is such a syzygy, namely,

$$f.(fl^2)^3 - l.(Hl) - (fl)^2.(fl) = 0.$$

For from $(f, (fl), l) = 0$ it follows that

$$f.((fl)^2 l) + (fl)^2 (lf) + l.(f, (fl)^2) = 0.$$

But the series $\begin{pmatrix} l & f & l \\ 1 & 0 & 2 \end{pmatrix}$ shows at once that $((fl)^2 l) = -(l^2 f)^3 = (fl^2)^3$, and the series $\begin{pmatrix} f & l & f \\ 0 & 1 & 2 \end{pmatrix}$ gives $((fl)^2 f) = ((ff)^2 l) = (Hl)$. Therefore

$$f.(fl^2)^3 - (fl)^2.(fl) - l(Hl) = 0.$$

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[*] $(i.m)$ denotes a covariant of degree i, order m.

[†] The symbol $\Delta(j, i; m)$ is used in the same sense as in Franklin, AMERICAN JOURNAL, Vol. III, p. 129.

In this case also, then, von Gall's table is correct and Sylvester's is defective by one groundform, owing to another exception to the Fundamental Postulate.

The case of next higher degree is of degorder 7.11. Von Gall shows one irreducible covariant $(Tl^2)^4$. There are 15 reducible covariants, namely

$$(1.7)(6.4)[2],\quad (2.2)(5.9)[2],\quad (2.6)(5.5)[2],\quad (2.10)(5.1),\quad (3.3)(4.8)[2],$$
$$(3.5)(4.6),\quad (3.7)(4.4)[2],\quad (3.11)(4.0),\quad (2.2)^2(3.7),\quad (2.2)(2.6)(3.3),$$

which are :

$$f.\begin{cases}(kl^2)^3\\(pl)\end{cases},\quad l.\begin{cases}(\gamma l)^2\\(\varepsilon l)\end{cases},\quad k.\begin{cases}g\\(fl^2)^3\end{cases},\quad H.(rl)^2,\quad r.\begin{cases}\Delta\\(Hl)\end{cases},$$
$$(fl)^2.(kl),\quad (fl).\begin{cases}p\\(kl)^2\end{cases},\quad \gamma.C,\quad l^2.(fl),\quad l.k.r.$$

The value of $\Delta(7, 7; 11)$ is 11, the number of linearly independent covariants of this degorder. There must therefore be certainly four independent syzygies among these reducible covariants, and five if $(Tl^2)^4$ is irreducible Conversely, $(Tl^2)^4$ is irreducible if there are five syzygies.

Perhaps the simplest proof of the irreducibility of $(Tl^2)^4$ is the following: If it be reducible it must satisfy a relation of the form $(Tl^2)^4 = \Sigma\, m\, K$, where K is a reducible covariant of degorder 7.11, and the m's are constants. This relation, for the same m's, must hold for all values of the coefficients of f. Take two sets of values for them:

$$\text{I.}\quad a_1 = a_2 = a_3 = a_5 = a_6 = 0,\qquad a_0 \neq 0,\quad a_4 \neq 0,\quad a_7 \neq 0.$$
$$\text{II.}\quad a_1 = a_2 = a_5 = a_6 = a_7 = 0,\qquad a_0 \neq 0,\quad a_3 \neq 0,\quad a_4 \neq 0.$$

On the basis of these suppositions the computations are not difficult. Of course only the first term of each covariant of degorder 7.11 is necessary. The supposition I gives, omitting forms and parts of forms that can not affect the result:

$$l = \binom{2}{1} a_0 a_7 xy + \binom{2}{2}(-20\, a_4^2)\, y^2,$$
$$l^2 = \binom{4}{2} 2/3\, a_0^2 a_7^2 x^2 y^2 + \binom{4}{3}(-20\, a_0 a_4^2 a_7)\, x y^3 + \binom{4}{4} 400\, a_4^4 y^4,$$
$$T = \binom{15}{1} 4/15\, a_0^2 a_4 x^{14} y + \binom{15}{4} 1/1365\, a_0^2 a_7 x^{11} y^4 + \text{etc.},$$
$$k = 2\, a_0 a_4 x^6 + \binom{6}{3} 1/10\, a_0 a_7 x^3 y^3 + \text{etc.},$$
$$p = 8\, a_0 a_4^3 x^4,$$
$$g = -\,2\, a_0^2 a_4^2 a_7 x^5 + \text{etc.},$$
$$(fl)^2 = -\,20\, a_0 a_4^2 x^5 + \text{etc.},$$
$$(fl) = a_0^2 a_7 x^7 + \text{etc.}$$

The supposition II gives:

$$l = -20\,a_3^2\,x^2 + \binom{2}{1}(-5\,a_3\,a_4)\,x\,y + \binom{2}{2}(-20\,a_4^2)\,y^2,$$

$$l^2 = 400\,a_3^4\,x^4 + \binom{4}{1}100\,a_3^3\,a_4\,x^3\,y + \binom{4}{2}150\,a_3^2\,a_4^2\,x^2\,y^2$$
$$+ \binom{4}{3}100\,a_3\,a_4^3\,x\,y^3 + \binom{4}{4}400\,a_4^4\,y^4,$$

$$T = a_0^2\,a_3\,x^{15} + \binom{15}{1}4/15\,a_0^2\,a_4\,x^{14}\,y + \binom{15}{3}(-31/91\,a_0\,a_3^2)\,x^{12}\,y^3$$
$$+ \binom{15}{4}(-80/273\,a_0\,a_3\,a_4)\,x^{11}\,y^4 + \text{etc.},$$

$$k = 2\,a_0\,a_4\,x^6 + \binom{6}{2}2\,a_3^2\,x^4\,y^2 + \binom{6}{3}3/2\,a_3\,a_4\,x^3\,y^3 + \binom{6}{4}2\,a_4^2\,x^2\,y^4,$$

$$p = (8\,a_0\,a_4^3 + 24\,a_3^4)\,x^4 + \binom{4}{1}6\,a_3^3\,a_4\,x^3\,y + \text{etc.},$$

$$\gamma = \binom{11}{1}10/11\,a_0\,a_3^2\,x^{10}\,y + \binom{11}{2}(-3/11\,a_0\,a_3\,a_4)\,x^9\,y^2 + \text{etc.},$$

$$\varepsilon = 2\,a_0\,a_3^2\,x^9 + \binom{9}{1}16/9\,a_0\,a_3\,a_4\,x^8\,y + \text{etc.},$$

$$r = -20\,a_3^3\,x^3 + \binom{3}{1}(-5\,a_3^2\,a_4)\,x^2\,y + \binom{3}{2}5\,a_3\,a_4^2\,x\,y^2 + 20\,a_4^3\,y^3,$$

$$(f\,l)^2 = -20\,a_0\,a_4^2\,x^5 + \text{etc.},$$

$$(f\,l) = -5\,a_0\,a_3\,a_4\,x^7 + \text{etc.},$$

$$H = \binom{10}{1}a_0\,a_3\,x^9\,y + \binom{10}{2}4/9\,a_0\,a_4\,x^8\,y^2 + \text{etc.},$$

$$g = 10\,a_0\,a_3\,a_4^3\,x^5 + \text{etc.}$$

From these may be obtained completely the numerical coefficients of the following terms of the first coefficients of all the covariants of degorder 7.11: $a_0^3\,a_4^3\,a_7$, $a_0^2\,a_3\,a_4^4$, $a_0\,a_3^5\,a_4$. Supposition I gives the first, II the others. The first two are sufficient for the purpose in hand. They are most plainly exhibited in a table.

	$(Tl^2)^4$	$f(kl^2)^3$	$f(pl)$	$l(\gamma l)^2$	$l(\varepsilon l)$	kg	$k(fl^2)^3$	$H(rl)^2$	$r\Delta$
$a_0^3\,a_4^3\,a_7$	$\frac{64}{3}$	-40	8	0	0	-4	-40	0	0
$a_0^2\,a_3\,a_4^4$	$\frac{880}{3}$	200	-40	0	0	20	200	0	0

	$r(Hl)^2$	$(fl)^2(kl)$	$(fl)p$	$(fl)(kl)^2$	$\gamma\,.\,C$	$l^2.(fl)$	lkr
$a_0^3\,a_4^3\,a_7$	0	-40	8	-40	0	0	0
$a_0^2\,a_3\,a_4^4$	0	200	-400	200	0	0	0

Here, except for the first form $(Tl^2)^4$, the second row is -5 times the first. It follows that in any expression $\Sigma m K$ the coefficient of $a_0^2 a_3 a_4^4$ will be -5 times that of $a_0^3 a_4^3 a_7$; but this is not true of $(Tl^2)^4$. Hence $(Tl^2)^4 \not\equiv \Sigma m K$ and $(Tl^2)^4$ is irreducible. In this case again, therefore, an exception to the Fundamental Postulate has made Sylvester's table incomplete.

A second proof consists in showing that there are five independent syzygies of degorder 7.11, which as already pointed out involves the irreducibility of $(Tl^2)^4$. These five syzygies will now be obtained.

In the syzygy $(f_1 f_2 f_3) = 0$ put $f_1 = (fl)^2$, $f_2 = k$, $f_3 = l$, and it becomes

$$(fl)^2 (kl) - k((fl)^2 l) + l((fl)^2 k) = 0.$$

But from $\begin{pmatrix} l & f & l \\ 1 & 0 & 2 \end{pmatrix}$ follows $((fl)^2 l) = (fl^2)^3$, and from $\begin{pmatrix} f & f & f \\ 0 & 3 & 4 \end{pmatrix}$ comes

$$(fk)^3 = 1/10\,(fl), \tag{1}$$

which, taken with $\begin{pmatrix} f & l & k \\ 0 & 1 & 2 \end{pmatrix}$, gives

$$((fl)^2 k) = (\gamma l)^2 + 10/11\,(\varepsilon l) + 1/45\,(fl)\,l.$$

These give the syzygy

$$l(\gamma l)^2 + 10/11\,l(\varepsilon l) - k(fl^2)^3 + (fl)^2 (kl) + 1/45\,l^2 (fl) = 0.$$

Again, if in the same syzygy $f_1 = (kl)^2$, $f_2 = f$, $f_3 = l$, the result is

$$(kl)^2 (fl) - f((kl)^2 l) + l((kl)^2 f) = 0.$$

Here $\begin{pmatrix} l & k & l \\ 1 & 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} k & l & f \\ 0 & 1 & 2 \end{pmatrix}$, together with (1), give

$$((kl)^2 l) = (kl^2)^3, \text{ and } ((kl)^2 f) = -(\gamma l)^2 + 12/11\,(\varepsilon l) - 1/30\,(fl)\,l.$$

These values make the syzygy become

$$-f(kl^2)^3 - l(\gamma l)^2 + 12/11\,l(\varepsilon l) + (fl)(kl)^2 - 1/30\,l^2 (fl) = 0.$$

Another syzygy results from the basic syzygy $[f_1 f_2 f_3 q]_3 = 0$ on putting $f_1 = f$, $f_2 = k$, $f_3 = q = l$, namely,

$$2(fk)(ll)^2 - 2(fl)(kl)^2 + 2(kl)(fl)^2 - l((fl)^2 k) + l((kl)^2 f) + l^2 (fk)^3 = 0,$$

which, on putting in the values of unknown transvectants already obtained, becomes

$$-l(\gamma l)^2 + 1/11\,l(\varepsilon l) + (fl)^2 (kl) - (fl)(kl)^2 + \gamma C + 1/45\,l^2 (fl) = 0.$$

A fourth syzygy comes from $(f_1 f_2 f_3) = 0$ on putting $f_1 = f$, $f_2 = (fr)^3$, $f_3 = l$. It is

$$f((fr)^3 l) - (fr)^3 (fl) - l((fr)^3 f) = 0.$$

From $\binom{f\,f\,f}{1\,3\,4}$ it is easy to find

$$(fl)^2 = -5(fk)^4, \tag{2}$$

and this, taken with $\binom{f\,f\,l}{0\,2\,4}$ and $\binom{f\,k\,f}{1\,3\,4}$, yields by elimination

$$(fr)^3 = 5/2\,p - 1/10\,l^2. \tag{3}$$

Moreover, on this account,

$$((fr)^3 l) = 5/2\,(pl), \tag{4}$$

since $(l^2, l) = 0$.

From $\binom{f\,k\,f}{0\,4\,3}$, $\binom{k\,f\,f}{1\,3\,3}$, $\binom{f\,k\,f}{0\,3\,4}$ and $\binom{f\,f\,f}{3\,3\,4}$ [the last shows

$$(fk)^6 = 0 \tag{5}$$]

elimination gives

$$(fr)^2 = 1/2\,(lk). \tag{6}$$

But this, in the result of elimination between $\binom{k\,l\,f}{0\,2\,1}$ and $\binom{k\,l\,f}{0\,1\,2}$, together with (1), gives

$$((fr)^2 f)^2 = -1/6\,(\gamma l)^2 - 7/22\,(\varepsilon l) + 1/45\,(fl)\,.\,l.$$

Now from $\binom{f\,r\,f}{0\,2\,2}$ and $\binom{f\,r\,f}{0\,1\,3}$ it is easy to obtain

$$((fr)^3 f) = 3((fr)^2 f)^2 - 1/2\,k\,.\,r,$$

which gives at once

$$((fr)^3 f) = -1/2\,(\gamma l)^2 - 21/22\,(\varepsilon l) + 1/15\,(fl)\,.\,l - 1/2\,k\,.\,r.$$

The syzygy then becomes

$$5/2\,f(pl) - 5/2\,p(fl) + 1/30\,l^2(fl) + 1/2\,l(\gamma l)^2 + 21/22\,l(\varepsilon l) \\ + 1/2\,k\,.\,r\,.\,l = 0.$$

The fifth syzygy is obtained from $\{f_1 f_2\, t q\}_4 = 0$ by putting $f_1 = f_2 = f$, $t = r$, $q = l$. It is

$$2f((fr)^3 l) - 3f((fl)^2 r)^2 - rl(ff)^4 - 3(ff)^2(rl)^2 + 3(fr)^2(fl)^2 \\ - 2(fl)(fr)^3 + 3r((fl)^2 f)^2 = 0.$$

From $\binom{f\ l\ r}{0\ 2\ 2}$ comes $((fl)^2 r)^2 = ((fr)^2 l)^2 + 1/3\,((fr)^3 l)$, and from $\binom{l\ l\ k}{1\ 2\ 0}$, $\binom{l\ k\ l}{1\ 1\ 1}$ and (6) comes

$$((fr)^2 l)^2 = -1/3\,(kl^2)^3. \tag{7}$$

Therefore $((fl)^2 r)^2 = -1/3\,(kl^2)^3 + 1/3\,((fr)^3 l)$.

Again, $\binom{f\ f\ l}{0\ 2\ 2}$ gives $((fl)^2 f)^2 = (Hl)^2 + 5/18\,k\,.\,l$.

These values, together with some of those previously obtained, give the fifth syzygy,

$$5/2 f(pl) + f(kl^2)^3 - 5p(fl) - 3H(rl)^2 + 3r(Hl)^2 - 3/2(kl)(fl)^2$$
$$- 1/6\,k\,.\,l\,.\,r + 1/5\,l^2(fl) = 0.$$

That these five syzygies are linearly independent may be ascertained by the usual tests or briefly thus: The terms $k(fl^2)^3$, $\gamma\,.\,C$, $H(rl)^2$ appear in the first, third and last syzygies respectively, and in no others. These syzygies, then, could enter into no linear relation connecting the five. The only remaining possibility is that the second and fourth syzygies should be related, which is clearly not the case.

There are then five linearly independent syzygies of degorder 7.11, and as already noted this involves the existence of a groundform and therefore of an exception to the Fundamental Postulate.

The next case is that of degorder 8.10. Von Gall shows one irreducible covariant $(\zeta l^2)^4$. There are 20 reducible covariants, namely:

$$(1.7)(7.3)\,[2],\ (2.2)(6.8)\,[2],\ (2.6)(6.4)\,[2],\ (3.3)(5.7)\,[2],\ (3.5)(5.5)\,[2],$$
$$(3.7)(5.3)\,[2],\ (3.9)(5.1),\ (4.0)(4.10),\ (4.4)(4.6)\,[2],\ (1.7)(3.3)(4.0),$$
$$(1.7)(2.2)(5.1),\ (2.2)(3.3)(3.5),\ (2.2)^2(4.6).$$

These are:

$$f\begin{cases}(fl^3)^5\\(gl)^2\end{cases},\ l\begin{cases}\beta\\(Hl^2)^3\end{cases},\ k\begin{cases}(kl^2)^3\\(pl)\end{cases},\ r\begin{cases}\eta\\(\varepsilon l)^2\end{cases},\ (fl)^2\begin{cases}g\\(fl^2)^3\end{cases},\ (fl)\begin{cases}(rl)\\(fl^2)^4\end{cases},$$
$$\varepsilon\,.\,(rl)^2,\ C\,.\,(Hl),\ \left.\begin{matrix}p\\(kl)^2\end{matrix}\right\}(kl),\ f\,.\,r\,.\,C,\ f\,.\,l\,.\,(rl)^2,\ l\,.\,r\,.\,(fl)^2,\ l^2(kl).$$

A computation of $\Delta(7, 8; 10)$ shows that there are fourteen linearly independent covariants of degorder 8.10. There must then certainly be six linearly independent syzygies of this degorder, and if there are seven there must be a groundform. As von Gall's system is certainly complete, it may be taken as the form $(\zeta l^2)^4$. There are seven such syzygies, as will now be shown.

In $(f_1 f_2 f_3) = 0$ let $f_1 = f$, $f_2 = (kr)^3$, $f_3 = l$; then,

$$f((kr)^3 l) - (kr)^3 . (fl) - l . ((kr)^3 f) = 0.$$

From $\binom{f\ f\ r}{0\ 3\ 4}$, $\binom{f\ f\ r}{0\ 2\ 5}$, $\binom{k\ l\ f}{0\ 5\ 1}$, $\binom{k\ l\ f}{0\ 4\ 2}$ and $\binom{l\ l\ f}{2\ 2\ 0}$, together with (5) and (6), elimination gives

$$(kr)^3 = 1/10\,(fl^2)^4 - 3/10\,(rl). \tag{8}$$

Therefore

$$((kr)^3 l) = 1/10\,((fl^2)^4 l) - 3/10\,((rl)\,l),$$

which, by the aid of $\binom{l\ r\ l}{0\ 1\ 1}$ and $\binom{l^2\ f\ l}{1\ 0\ 4}$, becomes

$$((kr)^3 l) = 1/10\,(fl^3)^5 - 1/5\,l . (rl)^2 + 3/20\,r . C. \tag{9}$$

Also

$$((kr)^3 f) = 1/10\,((fl^2)^4 f) - 3/10\,((rl) f),$$

which, by $\binom{f\ l^2\ f}{0\ 1\ 4}$ and $\binom{l\ l\ k}{0\ 1\ 0}$ and an elimination between $\binom{r\ l\ f}{0\ 1\ 1}$, $\binom{l\ r\ f}{0\ 1\ 1}$ and $\binom{f\ r\ l}{0\ 1\ 1}$, becomes

$$((kr)^3 f) = -1/20\,(rl)^2 . f + 17/120\,(kl) . l + 3/20\,(fl)^2 . r + 1/5\,(Hl^2)^3. \tag{10}$$

The syzygy becomes, then,

$$\begin{aligned}3/10\,(rl) . (fl) - 1/10\,(fl^2)^4 . (fl) - 3/20\,f . l . (rl)^2 - 17/120\,(kl) . l^2 \\ - 3/20\,(fl)^2 . r . l - 1/5\,l . (Hl^2)^3 + 1/10\,f(fl^3)^5 + 3/20\,f . r . C = 0.\end{aligned}$$

The second syzygy comes directly from $(f_1 f_2 f_3) = 0$ on putting $f_1 = p$, $f_2 = k$, $f_3 = l$. That is,

$$p . (kl) + k . (lp) + l . (pk) = 0,$$

or

$$p . (kl) - k . (pl) - l . \beta = 0.$$

The third syzygy is found from $(f_1 f_2 f_3) = 0$, when $f_1 = k$, $f_2 = (kl)^2$, $f_3 = l$. It is

$$k . ((kl)^2 l) - (kl)^2 . (kl) - l . ((kl)^2 k) = 0.$$

$((kl)^2 l)$ is evaluated at once by $\binom{l\ k\ l}{1\ 0\ 2}$, which gives $((kl)^2 l) = (kl^2)^3$.

$((kl)^2 k)$ is more complicated. First from $\binom{k\ l\ k}{0\ 1\ 2}$ comes $((kl)^2 k) = ((kk)^2 l)$.

Next express the groundform $(kk)^2$ differently. Namely $\begin{pmatrix} f & f & k \\ 0 & 3 & 3 \end{pmatrix}$, $\begin{pmatrix} f & l & f \\ 0 & 2 & 2 \end{pmatrix}$, $\begin{pmatrix} f & l & f \\ 0 & 3 & 1 \end{pmatrix}$, together with (1), (2) and (5), give

$$(kk)^2 = -7/45\, k\,.\,l - 1/5\,(Hl)^2 - 2/5\,(fr).$$

Then

$$((kk)^2\, l) = -7/45\,(kl, l) - 1/5\,((Hl)^2\, l) - 2/5\,((fr)\, l).$$

This, by the aid of $\begin{pmatrix} l & k & l \\ 0 & 1 & 0 \end{pmatrix}$, $\begin{pmatrix} l & H & l \\ 1 & 0 & 2 \end{pmatrix}$ and an elimination between $\begin{pmatrix} r & l & f \\ 0 & 1 & 1 \end{pmatrix}$, $\begin{pmatrix} l & r & f \\ 0 & 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} f & r & l \\ 0 & 1 & 1 \end{pmatrix}$, gives finally

$$((kl)^2 k) = ((kk)^2\, l) = -1/15\, l\,.\,(kl) - 1/5\,(Hl^2)^3 + 1/5\, f\,.\,(rl)^2 - 1/5\, r\,.\,(fl)^2,$$

and the third syzygy is

$$k\,.\,(kl^2)^3 - (kl)^2\,.\,(kl) + 1/15\, l^2\,.\,(kl) + 1/5\, l\,.\,(Hl^2)^3 - 1/5\, f\,.\,l\,.\,(rl)^2 + 1/5\, r\,.\,l\,.\,(fl)^2 = 0.$$

The fourth syzygy results from $[f_1 f_2 f_3 f_4]_2 = 0$ by placing $f_1 = f$, $f_2 = r$, $f_3 = f_4 = l$. Namely,

$$2\,(fl)\,(rl) + f\,.\,r\,.\,(ll)^2 + (fr)^2\,.\,l^2 - f\,.\,l\,.\,(rl)^2 - (fl)^2\,.\,r\,.\,l = 0.$$

Here it is only necessary to put $(fr)^2 = -1/2\,(kl)$ from (6) to get as the fourth syzygy

$$2\,(fl)\,.\,(rl) + f\,.\,r\,.\,C - 1/2\, l^2\,.\,(kl) - f\,.\,l\,.\,(rl)^2 - r\,.\,l\,.\,(fl)^2 = 0.$$

For the fifth sygyzy, in $[f_1 f_2 f_3\, q]_3 = 0$ put $f_1 = f$, $f_2 = (fl)^2$, $f_3 = q = l$. There results

$$2\,(f, (fl)^2)\,(ll)^2 - 2\,(fl)\,.\,((fl)^2\, l)^2 + 2\,((fl)^2\, l)\,(fl)^2 - l\,.\,((fl)^2, (fl)^2) + l\,(((fl)^2\, l)^2 f) + l^2\,(f, (fl)^2)^3 = 0.$$

The unknown transvectants here are evaluated thus: From $\begin{pmatrix} f & l & f \\ 0 & 1 & 2 \end{pmatrix}$ comes $(f, (fl)^2) = -(Hl)$. From $\begin{pmatrix} l & f & l \\ 0 & 2 & 2 \end{pmatrix}$ is found $((fl)^2\, l)^2 = (fl^2)^4$. $\begin{pmatrix} l & f & l \\ 1 & 0 & 2 \end{pmatrix}$ gives $((fl)^2\, l) = (fl^2)^3$. Obviously $((fl)^2, (fl)^2) = 0$. $(((fl)^2\, l)^2 f) = ((fl^2)^4 f)$ by means of $((fl)^2\, l)^2 = (fl^2)^4$ just obtained, and this with the aid of $\begin{pmatrix} f & l^2 & f \\ 0 & 1 & 4 \end{pmatrix}$ and $\begin{pmatrix} l & l & k \\ 0 & 1 & 0 \end{pmatrix}$ is easily changed to $(((fl)^2\, l)^2 f) = 2\,(Hl^2)^3 + 2/3\, l\,.\,(kl)$.

$\begin{pmatrix} f & l & f \\ 0 & 3 & 2 \end{pmatrix}$ at once shows that $(f, (fl)^2)^3 = -(kl)$.

These give for the fifth syzygy

$$2\,l\,.\,(Hl^2)^3 - 1/3\,l^2\,.\,(kl) + 2\,(fl)^2\,.\,(fl^2)^3 - 2\,(fl)\,.\,(fl^2)^4 - 2\,C\,.\,(Hl) = 0.$$

The sixth syzygy is obtained from $[f_1 f_2\, t q]_4 = 0$ by making $f_1 = f$, $f_2 = k$, $t = r$, $q = l$; namely,

$$\begin{gathered}-3f((kl)^2 r)^2 - 3\,k((fl)^2 r)^2 - 2\,r\,.\,l\,.\,(fk)^4 - 6\,(fk)^2\,.\,(rl)^2 + 3\,(fr)^2\,.\,(kl)^2 \\ + 3\,(fl)^2\,.\,(kr)^2 + 3\,r((fl)^2 k)^2 + 3\,r\,.\,((kl)^2 f)^2 + 2\,l\,.\,((fr)^3 k) \\ + 2\,.\,l\,.\,((kr)^3 f) = 0.\end{gathered}$$

To get $((kl)^2 r)^2$ in terms of known transvectants use $\begin{pmatrix} k & l & r \\ 0 & 2 & 2 \end{pmatrix}$. It gives, with the aid of (9),

$$((kl)^2 r)^2 = ((rk)^2 l)^2 + 1/25\,(fl^3)^5 - 2/25\,l\,.\,(rl)^2 + 3/50\,r\,.\,C.$$

From $\begin{pmatrix} f & l & r \\ 0 & 2 & 2 \end{pmatrix}$ and (3) and (7) is obtained the expression

$$((fl)^2 r)^2 = -1/3\,(kl^2)^3 + 5/6\,(pl).$$

$(fk)^4$ has already been found in (2), namely $(fk)^4 = -1/5\,(fl)^2$, and $(fr)^2$ also in (6) as $(fr)^2 = -1/2\,(kl)$.

From $\begin{pmatrix} f & l & k \\ 0 & 2 & 2 \end{pmatrix}$ and $\begin{pmatrix} l & f & l \\ 0 & 1 & 1 \end{pmatrix}$, together with (1) and (2), comes

$$((fl)^2 k)^2 = (\varepsilon l)^2 + 1/30\,(fl)^2\,.\,l - 2/45\,f\,.\,C.$$

To evaluate $((kl)^2 f)^2$ use $\begin{pmatrix} k & l & f \\ 0 & 2 & 2 \end{pmatrix}$, $\begin{pmatrix} l & f & l \\ 0 & 1 & 1 \end{pmatrix}$ and (1). Thus is found

$$((kl)^2 f)^2 = ((fk)^2 l)^2 - 1/6\,(fl)^2 l + 1/18\,C\,.f.$$

From (3), $((fr)^3 k) = 5/2\,pk - 1/10\,l^2 k$, which by use of $\begin{pmatrix} l & l & k \\ 0 & 1 & 0 \end{pmatrix}$ becomes

$$((fr)^3 k) = 5/2\,(pk) + 1/10\,l\,.\,(kl),$$

and from (10)

$$((kr)^3 f) = -1/20\,(rl)^2 f + 17/120\,(kl)\,.\,l + 3/20\,(fl)^2\,.\,r + 1/5\,(Hl^2)^3.$$

These substituted in the syzygy above serve to express it in transvectants of the complete system thus:

$$\begin{gathered}-3/25\,f\,.\,(fl^3)^5 - 3\,f\,.\,(gl)^2 - 11/75\,f\,.\,r\,.\,C + 7/50\,f\,.\,l\,.\,(rl)^2 - 5\,l\,.\,\beta \\ + 2/5\,l\,.\,(Hl^2)^3 + 3/10\,l\,.\,r\,.\,(fl)^2 + 29/60\,l^2\,(kl) + k\,.\,(kl^2)^3 - 5/2\,k\,.\,(pl) \\ + 6\,r\,.\,(\varepsilon l)^2 + 3\,(fl)^2\,.\,g - 6\,\varepsilon\,.\,(rl)^2 - 3/2\,(kl)^2\,.\,(kl) = 0.\end{gathered}$$

The seventh syzygy comes from making the same substitution, $f_1 = f$, $f_2 = k$, $t = r$, $q = l$, in $\{f_1 f_2 tq\}_4 = 0$. That makes it

$$\begin{gathered} 2k.((fr)^3 l) - 3k.((fl)^2 r)^2 - r.l.(fk)^4 - 3(fk)^2.(rl)^2 + 3(fr)^2.(kl)^2 \\ - 2(fl).(kr)^3 + 2r.((fl)^2 k)^2 + r.((kl)^2 f)^2 = 0. \end{gathered}$$

The value $((fr)^3 l) = 5/2(pl)$ was found in (4), and from (8)

$$(kr)^3 = 1/10\,(fl^2)^4 - 3/10\,(rl).$$

The other unknown transvectants were all evaluated for the preceding syzygy. Putting in the values there obtained and these two gives, for the final form of this syzygy,

$$\begin{gathered} -1/30 f.r.C + 1/10\, l.r.(fl)^2 + k(kl^2)^3 + 5/2\, k(pl) + 3r.(\varepsilon l)^2 \\ + 3/5\,(fl)(rl) - 1/5\,(fl)(fl^2)^4 - 3\varepsilon.(rl)^2 - 3/2\,(kl)^2.(kl) = 0. \end{gathered}$$

That these seven syzygies are independent appears from the fact that the determinant of the coefficients of the terms in $f(gl)^2$, $r.(\varepsilon l)^2$, $p.(kl)$, $(fl)^2(fl^2)^3$, $(kl)^2(kl)$, $f.(fl^3)^5$, $f.l.(rl)^2$ does not vanish. Other sets of terms are also easily found that give a non-vanishing determinant. In this case also, therefore, Sylvester's table is defective, owing to another exception to the Fundamental Postulate.

The last case of disagreement which will be taken up is that of degorder 10.4. Owing to its occurrence in a sense relatively early in the table of groundforms, it is comparatively simple. Sylvester shows here three irreducible covariants, while von Gall shows four. These are $(Hl^4)^7$, $(\beta l^2)^4$, $(\lambda l)^2$, (μl). There are the following ten reducible covariants of this degorder:

$$(2.2)(8.2)\,[3], \quad (3.3)(7.1)\,[3], \quad (4.0)(6.4)\,[2], \quad (5.1)(5.3)\,[2],$$

which are

$$l \left\{ \begin{array}{l} (kl^3)^5 \\ (pl^2)^3 \\ (\tau l) \end{array} \right. \qquad r\,.\left\{ \begin{array}{l} \alpha \\ (fl^3)^6 \\ (rl^2)^3 \end{array} \right. \qquad C\,.\left\{ \begin{array}{l} (kl^2)^3 \\ (pl) \end{array} \right. \qquad (rl)^2\,.\left\{ \begin{array}{l} (rl) \\ (fl^2)^4 \end{array} \right.$$

The value of $\Delta(7, 10; 4)$ is 13, which is therefore the number of linearly independent covariants of degorder 10.4. If then there exists any syzygy of this degorder among the ten reducible covariants enumerated, there must be $13 - 10 + 1 = 4$ groundforms, and as before the completeness of von Gall's system shows that the forms given may be taken as the four groundforms.

There is one such syzygy, namely,

$$r\,.\,(r\,l^2)^3 - l\,.\,(\tau\,l) - (r\,l)^2\,(r\,l) = 0.$$

For in $(f_1\,f_2\,f_4) = 0$ put $f_1 = (r\,l)^2$, $f_2 = r$, $f_3 = l$. Then

$$(r\,l)^2\,(r\,l) - r\,((r\,l)^2\,l) + l\,((r\,l)^2\,r) = 0.$$

But from $\begin{pmatrix} l & r & l \\ 1 & 0 & 2 \end{pmatrix}$ follows $((r\,l)^2\,l) = (r\,l^2)^3$, and from $\begin{pmatrix} r & l & r \\ 0 & 1 & 2 \end{pmatrix}$, $((r\,l)^2\,r) = ((r\,r)^2\,l) = (\tau\,l)$, which give the above syzygy at once.

Here also, therefore, another exception to the Fundamental Postulate makes Sylvester's result incorrect.

3. *Some Exceptions to the Fundamental Postulate for Quantics of Order Higher than the Seventh.*

As already pointed out there has existed a suspicion that the exceptions to the Fundamental Postulate might possibly be confined to the seventhic. Such is not the case, as will be shown in the present section. In fact the exceptions in the cases of higher orders are fairly numerous.

Every quantic of odd order as great as 5 has at least one syzygy of degorder 8.8, at least two of degorder 9.7, at least one of degorder 9.9, and at least one of degorder 10.4. These syzygies produce exceptions to the Fundamental Postulate in the cases of the ninethic degorders 8.8, 9.7, 10.4; eleventhic degorders 8.8, 9.7, 9.9, 10.4; thirteenthic degorders 8.8, 9.7, 9.9, 10.4.

These two facts are proven thus. Every quantic of odd order as great as 3 has a groundform of degorder 2.2, and one of degorder 3.3. For the quadratic has an invariant (2.0), and it follows from Hermite's Law of Reciprocity that every quantic of order $2m+1$ has a covariant of degorder 2.2, which is evidently irreducible for $m > 0$. Again, the cubic has an invariant (4.0) and a covariant (3.3). Then, by Hermite's Law, any quantic of order $4m+1$ or $4m+3$, *i. e.*, $2m+1$, has a covariant of degorder 3.3, which is also evidently irreducible for $m > 0$. Further, from the fact that the quadratic has a quadratic invariant it follows by Hermite's Law that every quantic of order $2m+3$ has a covariant of degorder 2.6, which is easily seen to be irreducible if $m > 0$. Again, the cubic has a (reducible) covariant of degorder 3.5, one of degorder 5.5 and a quartic invariant. Therefore a quantic of order $4m+3$ or $4m+5$, *i. e.*, $4m+1$, or in general $2m+1$, has a covariant of degorder 3.5, and this is irreducible if the order of the quantic is as great as 5.

For convenience, for a quantic of odd order in general, denote the groundform of degorder 2 2 by L, that of degorder 3.3 by R, that of degorder 2.6 by F, and that of degorder 3.5 by G.

If now, in $[f_1 f_2 q q']_4 = 0$, $f_1 = f_2 = F$, $q = q' = L$, it becomes a syzygy of degorder 8.8 for the quantic of odd order in general; namely,

$$2(FF)^2(LL)^2 + (FF)^4 L^2 - 2[(FL)^2]^2 - 4L((FL)^2 F)^2 + 2F.(FL^2)^4 = 0,$$

or, expressing the degorders involved,

$$2(4.8)(4.0) + (4.4)'(2.2)^2 - 2(4.4)^2 - 4(2.2)(6.6) + 2(2.6)(6.2) = 0,$$

which is a syzygy of degorder 8.8.

Two syzygies of degorder 9.7 are obtained thus. In $[f_1 f_2 f_3 q]_3 = 0$ put $f_1 = F$, $f_2 = R$, $f_3 = q = L$. This gives the syzygy

$$2(FR)(LL)^2 - 2(FL)(RL)^2 + 2(RL)(FL)^2 - L((FL)^2 R) + L((RL)^2 F)$$
$$+ L^2(FR)^3 = 0,$$

or

$$2(5.7)(4.0) - 2(4.6)(5.1) + 2(5.3)(4.4) - (2.2)(7.5) + (2.2)(7.5)'$$
$$+ (2.2)^2(5.3) = 0,$$

which is a syzygy of degorder 9.7. Again, in $[f_1 f_2 q q']_4 = 0$, the substitution $f_1 = F$, $f_2 = G$, $q = q' = L$ gives

$$2(FG)^2(LL)^2 + (FG)^4 L^2 - 2(FL)^2(GL)^2 - 2L((GL)^2 F)^2 - 2L((FL)^2 G)^2$$
$$+ F(GL^2)^4 + G(FL^2)^4 = 0,$$

or

$$2(5.7)(4.0) + (2.2)^2(5.3) - 2(4.4)(5.3) - 2(2.2)(7.5) - 2(2.2)(7.5)'$$
$$+ (2.6)(7.1) + (3.5)(6.2) = 0.$$

This is evidently independent of the preceding and makes therefore two syzygies of degorder 9.7.

To get the syzygy of degorder 9.9 put $f_1 = F$, $f_2 = G$, $f_3 = q = L$ in $[f_1 f_2 f_3 q]_3 = 0$. This gives

$$2(FG)(LL)^2 - 2(FL)(GL)^2 + 2(GL)(FL)^2 - L((FL)^2 G) + L((GL)^2 F)$$
$$+ L^2(FG)^3 = 0,$$

or

$$2(5.9)(4.0) - 2(4.6)(5.3) + 2(5.5)(4.4) - (2.2)(7.7) + (2.2)(7.7)'$$
$$+ (2.2)^2(5.5) = 0.$$

The syzygy of degorder 10.4 may be obtained from $(f_1 f_2 f_3) = 0$ by making $f_1 = (RL)^2$, $f_2 = R$, $f_3 = L$, or directly from the syzygy found for the seventhic case 10.4. For the result found there depends only on the degorders of l and r, and not at all on the order of the quantic. But the degorders of L and R are precisely those of l and r. Then it gives a syzygy true for the general quantic of odd order to replace l by L and r by R. This gives

$$R(RL^2)^3 - L((RR)^2 L) - (RL)^2 (RL) = 0,$$

or

$$(3.3)(7.1) - (2.2)(8.2) - (5.1)(5.3) = 0,$$

which is the general syzygy of degorder 10.4.

It now remains to show that these syzygies produce exceptions to the Fundamental Postulate. Sylvester's table for the ninethic shows two groundforms of degorder 8.8, two of degorder 9.7, and fourteen of degorder 10.4, and these numbers are certainly minima. In these three cases therefore, since syzygies of these degorders have just been found, there exist together groundforms and fundamental syzygies, *i. e.*, exceptions to the Fundamental Postulate.

Further, a computation* of the groundforms of the eleventhic and the thirteentic by the method of tamisage shows a positive number of them for degorders 8.8, 9.7, 9.9 and 10.4 in both cases. There are therefore exceptions to the Fundamental Postulate in all these cases, in fact double exceptions for degorder 9.7.

So far all exceptions to the Fundamental Postulate have been confined to quantics of odd order. It is the purpose of the present paragraph to show that

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* As the author is not aware where such results as these are to be found if they have been previously computed, he ventures to append them as far as he has found it desirable to carry them. They were calculated by "crude" or "brute" tamisage, which for a limited field costs less labor than the more complete reduction of the Generating Function to the Representative Form. The minus (written above for compactness) indicates syzygies.

TABLES OF GROUNDFORMS, AS GIVEN BY TAMISAGE, TO DEGREE 10, ORDER 10.

Eleventhic.

	0	1	2	3	4	5	6	7	8	9	10
1											
2			1				1				1
3				1		1		1		2	
4	2				3		2		4		3
5		2		4		5		7		7	
6			7		7		11		12		11
7		8		12		18		18		17	
8	10		17		25		28		19		16
9		21		37		37		24		7	
10	13		45		52		19		$\overline{8}$		$\overline{104}$

Thirteenthic.

	0	1	2	3	4	5	6	7	8	9	10
1											
2			1				1				1
3				1		1		1		2	
4	2				4		2		5		4
5		3		5		7		9		11	
6			11		11		18		20		20
7		14		21		32		36		39	
8	19		33		51		62		54		58
9		45		87		94		84		62	
10	33		112		145		100		57		$\overline{100}$

Fourteenthic.

	0	2	4	6	8	10
1						
2	1		1		1	
3		1		2	1	2
4	2		4	2	5	4
5		5	6	9	10	12
6	7	6	15	19	22	25
7	4	22	31	37	47	42
8	18	36	58	74	67	75
9	23	73	108	119	112	67
10	52	129	179	175	72	$\overline{40}$

there are exceptions to the Postulate in the case of quantics of even order. It will be shown first, that there is in general a syzygy of degorder 10.6 for every quantic of order $4m+2\ [m>0]$, and second, that this produces exceptions to the Fundamental Postulate in the cases of the tenthic and of the fourteenthic. The cubic has an invariant of degree 4 and a covariant of degorder 2.2. It follows by Hermite's Law that any quantic of order $4m+2$ has a covariant (3.2), and this is necessarily irreducible if $m>0$. Similarly, since the quadratic has a quadratic invariant, every quantic of even order has a covariant (2.4). Denote the covariant (3.2) by Q and (2.4) by P. Now in $(f_1 f_2 f_3)=0$ put $f_1=(PQ)^2$, $f_2=P$, $f_3=Q$. Then

$$(PQ)^2(PQ)-P((PQ)^2Q)+Q((PQ)^2P)=0;$$

or, since from $\begin{pmatrix} Q & P & Q \\ 1 & 0 & 2 \end{pmatrix}$, $((PQ)^2Q)=(PQ^2)^3$, and from $\begin{pmatrix} P & Q & P \\ 0 & 1 & 2 \end{pmatrix}$, $((PQ)^2P)=((PP)^2Q)$, this may be written, perhaps a little more neatly, as

$$(PQ)^2(PQ)-P(PQ^2)^3+Q((PP)^2Q)=0,$$

or, expressing the degorders involved,

$$(5.2)(5.4)-(2.4)(8.2)+(3.2)(7.4)=0,$$

which is a syzygy of degorder 10.6.

The tenthic is a special case of this. Sylvester's table for the tenthic shows ten groundforms of degorder 10.6. There is therefore an exception to the Fundamental Postulate in this case. Again, the method of tamisage applied to the fourteenthic gives 175 groundforms of degorder 10.6 (see foot-note, p. 65), so that here too is an exception.

It is easy to show that the twelfthic has at least one exception to the Fundamental Postulate. For it has groundforms (2.4) and (2.8), as may be readily ascertained by Hermite's Law or by reference to Sylvester's table for the twelfthic.* Let the quantic be represented by f, the covariant (2.4) by u and (2.8) by v. Then make $f_1=f$, $f_2=u$, $f_3=v$, in $(f_1 f_2 f_3)=0$. That is,

$$f(uv)+u(vf)+v(fu)=0,$$

or

$$(1.12)(4.10)+(2.4)(3.18)+(2.8)(3.14)=0,$$

which is a syzygy of degorder 5.22. Now Sylvester's table (*loc. cit.*) shows four groundforms of degorder 5.22. Consequently there is here also an exception to the Postulate.

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* Sylvester, AMERICAN JOURNAL, Vol. IV, p. 48.

With regard to the syzygies in this section, where some forms occur which might be reducible, one precaution is necessary, namely, to be sure that the syzygies are actually such, not merely identities. Take first the syzygy of degorder 8.8 belonging to quantics of odd order. As already pointed out, the forms F and L are irreducible. So also is the form $(LL)^2$, for it is of degorder 4.0 and no quantic of odd order can have a quadratic invariant. Therefore the only way in which this syzygy might reduce to an identity would be

$$2(2.2)(2.6)(4.0)+(4.4)'(2.2)^2-2(4.4)^2-4(2.2)(2.6)(4.0)$$
$$+\,2(2.6)(2.2)(4.0)=0.$$

But this requires

$$(4.4)'(2.2)^2=2(4.4)^2,$$

or, since the right-hand member is a square (barring the numerical constant) and so is the second factor on the left, $(4.4)'$ must be a square. But since there is but one form (2.2), namely L, this gives $(4.4)'$ (and hence (4.4) also) = a numerical multiple of L^2. Referring back to the transvectant values of these symbols, this is

$$(FF)^4=\lambda_1\,.\,L^2,\ \text{and}\ (FL)^2=\lambda_2\,.\,L^2$$

(where λ_1 and λ_2 are numerical multiples), which are not true if the order of the quantic is as great as 5. The syzygy then can not reduce to an identity.

The syzygies of degorder 9.7 belonging to quantics of odd order are simpler. In the first, the second term (4.6)(5.1) can not be reduced. For the factor (4.6) is irreducible because a quantic of odd order can not have a form (2.4) nor (3.1). (This appears at once from Hermite's Law applied to the quadratic and cubic respectively.) (5.1) is also plainly irreducible. This term then can not go out identically and this is an actual syzygy. Similarly in the second syzygy of this degorder the next to the last term (2.6)(7.1) is plainly irreducible, and therefore the syzygy can not reduce to an identity.

In the syzygy of degorder 9.9 for quantics of odd order the second term (4.6)(5.3) is irreducible into factors of lower degorder, and can not cancel; so the syzygy can not be an identity.

In the same way the last syzygy for odd quantics, of degorder 10.4, is easily seen to be an actual syzygy.

The syzygy proven for quantics of order $4m+2$ was

$$(5.2)(5.4)-(2.4)(8.2)+(3.2)(7.4)=0.$$

The quantic of order $4m+2$ can not have a cubic invariant, for the cubic has only one invariant, degree 4, so by Hermite's Law only a quantic of order $4m$

can have a cubic invariant. Also a quantic of even order can not have a covariant (2.2) nor one of degorder 4.2. The possible decompositions, then, of the above relation are:

$$\left.\begin{array}{r}(2.4)\,[(5.0)\,(3.2)]\\ \text{or}\;\;(2.4)\,[(2.0)\,(6.2)]\end{array}\right\} = \left\{\begin{array}{l}\phantom{\text{or}\;\;}(3.2)\,[(5.0)\,(2.4)] + [(3.2)\,(2.0)]\,(5.4)\\ \text{or}\;\;(3.2)\,[(2.0)\,(5.4)] + [(3.2)\,(2.0)]\,(5.4)\end{array}\right.$$

In any one of the four possible combinations of the two members it is to be understood that factors included in brackets [] may occur as a single irreducible one, and that a reduction into factors may introduce numerical multiples which are not shown. It may be remarked that the first possibility on each side can occur only if the order of the quantic is $4m + 18\lambda\,[m \geqq 0,\ \lambda > 0]$. For since the quintic has irreducible invariants of degrees 4, 8, 12, 18, only, to have a quintic invariant a quantic must be of order $4\lambda_1 + 8\lambda_2 + 12\lambda_3 + 18\lambda_4$, *i. e.*, of order $4m + 18\lambda$. This number can not coincide with $4m + 2$, the order of the quantic under discussion, unless $\lambda > 0$. However, in any case the syzygy can not reduce to an identity.

The syzygy found for the twelfthic is also obviously not an identity.

4. *Conclusion.*

It appears then that exceptions to the Fundamental Postulate of Tamisage are much more widespread than the few cases pointed out by Hammond. They are not confined to the seventhic, nor to quantics of odd order. Indeed, if we divide quantics into three classes, according as the orders are $4m$, $4m + 2$, or $2m + 1$, we may say very roughly that in the tables of groundforms and syzygies, there is a region of syzygies that is nearly fixed; *i. e.*, that changes but little as the order of the quantic is increased within the class. (For odd quantics this region is likely to begin when the sum of the degree and order exceeds 14.) On the other hand, the region of groundforms in the tables appears to extend continuously as the order of the quantic grows greater in its class. When these two regions overlap, exceptions to the Fundamental Postulate occur. This begins to be the case for odd quantics when the order is as great as 7, for even quantics when it is as great as 10. There seems to be no reason for supposing that Sylvester's method will ever become exact again for quantics of a sufficiently high finite order. Sylvester's method remains valuable only as furnishing a minimum number of groundforms. It is likely to be exact for a quantic of high order only when the sum of the degree and the order of the case in question is relatively small.

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